

GENERIC PROPERTIES OF CLOSED ORBITS FOR LAGRANGIAN FLOWS ON SURFACES

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ABSTRACT. We prove a local perturbation theorem for the k -jets of the Poincaré map over a closed orbit of the flow of a Lagrangian system $L : TM \rightarrow \mathbb{R}$ on a closed surface M . The perturbations consists of adding to the Lagrangian L a C^∞ -potential $u : M \rightarrow \mathbb{R}$. Therefore we obtain generic properties of closed orbits in the sense of Mañé.

1. INTRODUCTION.

Let $L : TM \rightarrow \mathbb{R}$ be a smooth Tonelli Lagrangian defined in a closed smooth manifold M , i.e., L satisfy the two conditions: *convexity*: for each fiber $T_x M$, the restriction $L(x, v)$ has positive defined Hessian, and *superlinearity*: $\lim_{\|v\| \rightarrow \infty} \frac{L(x, v)}{\|v\|} = \infty$ uniformly in $x \in M$. In this paper, we assume also that M has dimension two.

The *action* of L over an absolutely continuous curve $\gamma : [a, b] \rightarrow M$ is defined by:

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt.$$

The extremal curves of the action are given by solutions of the *Euler-Lagrange equations* that in local coordinates can be written as:

$$(1) \quad \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} = 0.$$

Since L is convex and M is compact, the Euler-Lagrange equations define a complete flow $\phi_t^L : TM \rightarrow TM$, that is called the *Lagrangian flow of L* and is defined by

$$\phi_t^L(x_0, v_0) = (\gamma(t), \dot{\gamma}(t)),$$

where $\gamma : \mathbb{R} \rightarrow M$ is the solution of (1) with initial conditions $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v_0$.

Let ω_0 be the canonical symplectic structure in the cotangent bundle T^*M . Then the Lagrangian flow of L is conjugated to a Hamiltonian flow in (T^*M, ω_0) by the Legendre transformation $\mathcal{L} : TM \rightarrow T^*M$, defined by:

$$\mathcal{L}(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v) \right).$$

The corresponding Hamiltonian $H : T^*M \rightarrow \mathbb{R}$, is given by:

$$(2) \quad H(x, p) = \max_{v \in T_x M} \{p(v) - L(x, v)\} = E_L(\mathcal{L}^{-1}(x, p)),$$

where $E_L : TM \rightarrow \mathbb{R}$ is the *Energy function*, that is defined as

$$E_L(x, v) = \frac{\partial L}{\partial v}(x, v) \cdot v - L(x, v).$$

Given $c \in \mathbb{R}$, the energy level $E_L^{-1}(c) \subset TM$ is compact and invariant by the Lagrangian flow. Suppose that $\theta_t = \phi_t^L(\theta)$ is a closed (or periodic) orbit with period $T_0 > 0$ in $E_L^{-1}(c)$ and let $\Sigma \subset E_L^{-1}(c)$ be a local transversal section in the energy level $E_L^{-1}(c)$ at the point θ . We say that θ_t is *nondegenerate* if the linearized Poincaré map $d_\theta P = d_\theta P(\theta, \Sigma, L) : T_\theta \Sigma \rightarrow T_\theta \Sigma$ does not admit a root of unity as eigenvalue. We say that θ_t is *hyperbolic* if $d_\theta P$ does not have eigenvalues with norm equal to 1, and that θ_t is *elliptic* if all the eigenvalues of $d_\theta P$ have norm one but they are not roots of unity. For surfaces, a nondegenerate closed orbit is either elliptic or hyperbolic. Given two hyperbolic periodic orbits θ_t and η_t of the Lagrangian flow ϕ_t^L , a *heteroclinic orbit* from θ_t to η_t is an orbit whose α -limit is θ_t and its ω -limit is η_t . The *strong stable* and *strong unstable manifolds* of the hyperbolic periodic orbit θ_t at the point $\theta \in \theta_t$ are defined as

$$W^{ss}(\theta) = \{v \in E_L^{-1}(c); \lim_{t \rightarrow \infty} d(\theta_t, \phi_t^L(v)) = 0\}, \text{ and}$$

$$W^{su}(\theta) = \{v \in E_L^{-1}(c); \lim_{t \rightarrow -\infty} d(\theta_t, \phi_t^L(v)) = 0\}$$

respectively. The *(weak) stable* and *(weak) unstable manifolds* of the hyperbolic periodic orbit θ_t are defined as

$$W^s(\theta_t) = \bigcup_{t \in \mathbb{R}} \phi_t^L(W^{ss}(\theta)) \text{ and } W^u(\theta_t) = \bigcup_{t \in \mathbb{R}} \phi_t^L(W^{su}(\theta))$$

respectively. The sets $W^s(\theta_t)$ and $W^u(\theta_t)$ are ϕ_t^L -invariant and they are immersed submanifolds of $E_L^{-1}(c)$. A heteroclinic orbit from θ_t to η_t is in the intersection $W^s(\theta_t) \cap W^u(\eta_t)$. When this intersection is transversal in $E_L^{-1}(c)$ we say that the heteroclinic orbit is *transversal*.

Let us recall some facts about the jet space for symplectic maps in $(\mathbb{R}^{2n}, \omega_0 = dx \wedge dy)$. Let $Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$ be the space of smooth symplectic diffeomorphisms $f : (\mathbb{R}^{2n}, \omega_0) \rightarrow (\mathbb{R}^{2n}, \omega_0)$, such that $f(0) = 0$. Given $k \in \mathbb{N}$, consider the equivalence relation \sim_k in $Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$, defined as:

$$f \sim_k g \Leftrightarrow \text{the Taylor polynomials of degree } k \text{ in zero are equal.}$$

We define the *k-jet* of $f \in Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$ that we denote by $j^k(f) = j^k(f)(0)$, as the equivalence class of f with respect to the relation \sim_k . The *space of symplectic k-jets* $J_s^k(n)$ is the set of all equivalence classes with respect to the relation \sim_k of elements of $Diff_{\omega_0}(\mathbb{R}^{2n}, 0)$. Observe that $J_s^k(n)$ is a vector space that is also a Lie group, if we consider the product defined by

$$j^k(f) \cdot j^k(g) = j^k(f \circ g), \quad \forall f, g \in Diff_{\omega_0}(\mathbb{R}^{2n}, 0).$$

When $k = 1$, we can identify $J_s^1(n)$ with the classic Lie group $Sp(n)$. We say that a subset $Q \subset J_s^k(n)$ is *invariant* if

$$(3) \quad \sigma \cdot Q \cdot \sigma^{-1} = Q, \quad \forall \sigma \in J_s^k(n).$$

Note that if $\theta_t = \phi_t^L(\theta)$ is a periodic orbit in some energy level $E_L^{-1}(c)$ and $\Sigma \subset E_L^{-1}(c)$ is a local transversal section at the point θ , then the symplectic form $\mathcal{L}^*\omega_0$ on TM induces a symplectic form on Σ and the Poincaré map $P(\theta, \Sigma, L) : \Sigma \rightarrow \Sigma$ becomes a symplectic diffeomorphism. Therefore, using Darboux coordinates, we can assume that $j^k(P(\theta, \Sigma, L)) \in J_s^k(1)$. Given an invariant subset $Q \subset J_s^k(1)$ and a closed orbit θ_t , it follows from (3) that the property: the k-jet the Poincaré map over θ_t belongs to Q , is well defined, because it is independent of the section Σ and the coordinate system.

In this paper we will consider generic properties in the sense of to R. Mañé. In [Mañ96], he shows “how the theory of minimizing measures becomes much stronger and more accurate if we restrict it to generic Lagrangians”. Let $C^\infty(M)$ be the space of smooth functions $u : M \rightarrow \mathbb{R}$ endowed with the C^∞ topology. Recall that a subset $\mathcal{O} \subset C^\infty(M)$ is called *residual* if it contains a countable intersection of open and dense subsets. We say that a *property is generic* (in the sense of Mañé), if

for each Lagrangian L , there exists a residual subset $\mathcal{O} \subset C^\infty(M)$, such that the property holds for all modified Lagrangians $L - u$, $u \in \mathcal{O}$.

In [Oli08], E. Oliveira proves a conservative version of the Kupka-Smale Theorem for generic Lagrangians on surfaces. More precisely, he proves that, if in the configuration space M has dimension two, for each $c \in \mathbb{R}$, there exists a residual set $\mathcal{O} = \mathcal{O}(c) \subset C^\infty(M)$, such that, every Lagrangian $L - u$, $u \in \mathcal{O}$ satisfies:

$$P_{K-S} : \begin{cases} (i) & E_L^{-1}(c) \text{ is a regular energy level,} \\ (ii) & \text{all closed orbits in } E_L^{-1}(c) \text{ are either hyperbolic or elliptic} \\ (iii) & \text{all heteroclinic intersections in } E_L^{-1}(c) \text{ are transversal.} \end{cases}$$

The main goal of the present paper is to extend this result to include conditions on the higher order derivatives of the Poincaré maps of closed orbits. It is motivated by the fact that some important proprieties of the dynamical behavior near elliptic closed orbits depend on the higher order derivatives of the corresponding Poincaré map.

The extension that we prove here is analogous to what has been done for other classes of conservative systems. Let us consider, for instance, generic properties of closed geodesics, and recall that a bumpy metric is a metric such that all closed geodesics are non-degenerated. In this case the subset of bumpy metrics in the space of smooth Riemannian metrics on M is residual. This theorem is attributed to R. Abraham [Abr70]; also see D.V. Anosov [Ano82], where a complete proof is given. In [KT72], Klingenberg and Takens extend the bumpy metric theorem to include conditions on the k -jets of the Poincaré map over closed orbits for geodesic flows. For the class of magnetic flows on surfaces, a complete study of generic properties of closed orbits can be seen in [Mir06].

Using the notation above, we can state our local perturbation theorem.

Theorem 1. *Let $Q \subset J_s^k(1)$ open and invariant, such that $j^k(P(\theta, \Sigma, L)) \in \overline{Q}$. Then there exists a smooth potential $u : M \rightarrow \mathbb{R}$, arbitrarily C^r -close to zero, with $r > k$, such that*

- θ_t is also a closed orbit of the Lagrangian flow for $L - u$ and
- $j^k(P(\theta, \Sigma, L - u)) \in Q$.

Combining the Kupka-Smale Theorem and the above theorem, we obtain:

Theorem 2. *Let M be a closed two dimensional manifold. Given an open and dense invariant subset $Q \subset J_s^k(1)$ and $c \in \mathbb{R}$, the property*

$$P_Q : \text{the } k\text{-jet of the Poincaré map of every closed orbit in } E_L^{-1}(c) \text{ belongs to } Q$$

is generic, in the sense of Mañé, for Lagrangian systems on M .

We would like to point out that we don't know if the above properties are generic for Lagrangians on manifolds of arbitrary dimensions. In the proof of Theorem 1, we shall use the local perturbation result in [Oli08, Theorem 4.5] which was proved only in dimension two.

As an application of Theorem 1, we obtain a result about the topological entropy of Lagrangian flows. The *topological entropy* is a dynamical invariant that, roughly speaking, measures its orbit structure complexity. Its precise definition can be found in [Bow75]. The relevant question about the topological entropy is whether it is positive or vanishes. It is well known that if a flow contains a transversal homoclinic orbit, then it has positive topological entropy.

Proposition 3. *Let M be a closed two dimensional manifold. Suppose that the Lagrangian flow $\phi_t^L : TM \rightarrow TM$ has a non-hyperbolic closed orbit in a energy level $E_L^{-1}(c)$. Then, there is a potential function $u : M \rightarrow \mathbb{R}$ of norm arbitrarily small in the C^∞ -topology, such that the perturbed Lagrangian flow ϕ_t^{L-u} restricted to $E_{L-u}^{-1}(c)$ has positive topological entropy.*

In order to prove this proposition, we apply the Theorem 1 to approximate L by $L - u$ such that the Poincaré map of the non-hyperbolic orbit becomes a generic exact twist map in a small neighborhood of the elliptic fixed point. Then a result of Le Calvez [LC91] implies that this twist map has homoclinic orbits, it implies positive topological entropy. In particular, the flow has infinite many closed orbits. The details of this arguments will be given in section 4.

2. THE LOCAL PERTURBATION OF THE K-JET.

In other to prove the Theorem 1 we take the Hamiltonian point of view. Then, given the closed orbit $\theta_t = (\gamma(t), \dot{\gamma}(t))$ of the Lagrangian flow, let $\Gamma(t) = (\gamma(t), p(t))$ be the corresponding closed orbit of the Hamiltonian flow, defined by (2). Recall that, by the Kupka-Smale Theorem we can suppose that Γ is nondegenerate and the energy level $H^{-1}(c)$ that contain Γ is regular. Observe that perturbations $L - u$ of L are equivalent to perturbations of the kind $H + u$, where $u : M \rightarrow \mathbb{R}$ is a smooth function.

2.1. \mathbf{k} -General family of linear symplectic maps. First, we produce a perturbation of the 1-jet of the Poincaré map along the periodic orbit to put it in a particular position.

Let $\mathbb{R}[x, y]^k$ be the space of real homogeneous polynomials of degree k in the variables x, y . We fix the polynomial $F(x, y) := x^k$. For each $k \in \mathbb{N}$, let G_k be the set defined as:

$$G_k = \left\{ (\sigma_1, \dots, \sigma_k) \in Sp(1)^k; \{F(x, y), F(\sigma_1(x, y)), \dots, F(\sigma_k(x, y))\} \text{ is a basis for } \mathbb{R}[x, y]^k \right\}.$$

Definition 4. A one parameter family $\sigma : [a, b] \rightarrow Sp(1)$, with $\sigma(a) = \sigma_b = I$, is **\mathbf{k} -general**, if there exist times $t_1, \dots, t_k \in (a, b]$ such that $(\sigma_{t_1}, \dots, \sigma_{t_k})$ is a element of the set G_k .

First we observe that:

Proposition 5. For each $k \in \mathbb{N}$, the subset G_k is open and dense in $Sp(1)^k$.

Proof. Consider the one parameter family of symplectic matrix $\sigma : [0, 1] \rightarrow Sp(1)$, such that:

$$\sigma_t(x, y) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}.$$

It is easy to see that there exist values $t_1, \dots, t_k \in (0, 1]$ for what the polynomials $F(x, y)$ and $F(\sigma_{t_n}(x, y)) = (x + t_n y)^k$, with $n = 1, \dots, k$, form a basis of $\mathbb{R}[x, y]^k$. By fixing a basis in $\mathbb{R}[x, y]^k$, we have that G_k is the complement in $Sp(1)^k$ of the set:

$$G_k^c = \left\{ (\sigma_1, \dots, \sigma_k) \in Sp(1)^k; \det [F(\sigma_i)]_{0 \leq i \leq k} = 0 \right\}.$$

Therefore G_k is a non empty complementary set of a algebraic subset. This implies that G_k is open and dense in $Sp(1)^k$. \square

Now, let $T > 0$ be such that the arc $\gamma((0, T]) \subset M$ has not self-intersection points of γ , and take $W \subset M$ a tubular neighborhood of $\gamma([0, T])$, with $W \cap \gamma \subset \gamma([0, T])$. We consider

$$\mathcal{F}^1 = \mathcal{F}^1(\gamma, T, W) = \{u \in C^\infty(M); j^1(u)(\gamma([0, T])) \equiv 0 \text{ and } \text{Supp}(u) \subset W\}.$$

Note that, for each $u \in \mathcal{F}^1$, the curve $\Gamma(t)$ is also a closed orbit of the Hamiltonian flow $\psi_t^{H+u} : T^*M \rightarrow T^*M$ and that the energy levels $H^{-1}(c)$ and $(H + u)^{-1}(c)$ are tangent along of the curve $\Gamma(t)$ (see also remark 9). Choose $t_0 \in (0, T)$, and let $\mathcal{N}(t_0)$ be the linear subspace of $T_{\Gamma(t_0)}(T^*M)$ that is the symplectic orthogonal of the Hamiltonian vector field $X^H(\Gamma(t_0))$. Then, we can consider the map

$$\mathcal{S}_{t_0}^1 : \mathcal{F}^1 \rightarrow Sp(1)$$

defined by $\mathcal{S}_{t_0}^1(u) = d_{\Gamma(0)}P_{t_0}(u)$, where $d_{\Gamma(0)}P_{t_0}(u) : \mathcal{N}(0) \rightarrow \mathcal{N}(t_0)$ denotes the derivative of the Poincaré map $P_{t_0} = P(\Gamma(0), \Sigma(0), \Sigma(t_0), H + u)$, for the transversal sections $\Sigma_t \subset H^{-1}(c)$ at $\Gamma(t)$, such that $T_{\Gamma(t)}\Sigma_t = \mathcal{N}(t) := d_{\Gamma(t_0)}\psi_{t_0-t}^{H+u} \cdot \mathcal{N}(0)$, for $t \in [0, t_0]$.

Remark 6. *It was proved in [Oli08], that the derivative at zero of the map $\mathcal{S}_{t_0}^1 : \mathcal{F}^1 \rightarrow Sp(1)$ is onto in the Lie algebra $\mathfrak{sp}(1)$ of the classical Lie group $Sp(1)$. This implies that $\mathcal{S}_{t_0}^1$ is an open map in a neighborhood of zero. Therefore we can found a potential $u : M \rightarrow \mathbb{R}$, arbitrarily C^∞ -close to zero, that is adapted to each symplectic matrix in a small enough open ball centered at $\mathcal{S}_{t_0}^1(0) = d_{\Gamma(0)}P_{t_0}(H)$. Moreover the support of u can be choose arbitrarily small. See the proof of Theorem 4.5 in [Oli08] for the details.*

Combining its remark and Proposition 5, we obtain:

Lemma 7. *For each integer $k > 2$, there exists a smooth potential $u_0 : M \rightarrow \mathbb{R}$, with C^∞ -norm arbitrarily small such that $\Gamma(t)$ is also a closed orbit of the perturbed Hamiltonian flow $\psi_t^{H+u_0}$ and such that the one parameter family $t \mapsto d_{\Gamma(0)}P_t(H + u_0)$, for $t \in [0, T]$, is k -general.*

Proof. Given $k \in \mathbb{N}$, we set $t_0 = t_0(k, T) \in (0, T]$ such that $T = (k + 1)t_0$. We divide γ in $k + 1$ segments $\gamma_i : [0, t_0] \rightarrow M$ given by $\gamma_i(t) := \gamma(t + it_0)$, with $0 \leq i \leq k$. Consider the map

$$\mathcal{S} : \mathcal{F}^1 \rightarrow Sp(1)^k$$

defined by

$$\mathcal{S}(u) = (d_{\Gamma(0)}P_{t_0}(u), \dots, d_{\Gamma(0)}P_{kt_0}(u)).$$

Then, by Remark 6, each component of the map \mathcal{S} is a local submersion near $0 \in \mathcal{F}^1$. Since the $G^k \subset Sp(1)^k$ is dense (Proposition 5), we can choose a potential $u = u_1 + \dots + u_k \in \mathcal{F}^1$, with $\text{Supp}(u_i) \cap \gamma_j = \emptyset$ for $i \neq j$ and the C^∞ -norm arbitrarily small such that the one parameter family of linear symplectic maps associated to the linearized Poincaré map of the perturbation $H + u$ is k -general. \square

2.2. Perturbation of the k -jet. Let us now fix a local coordinate system in a neighborhood of a segment of the closed orbit $\Gamma(t)$ that we will use to give a local description of the Hamiltonian flow and to define an appropriated perturbation space. Let T_0 the minimal period of $\Gamma(t)$. We choose $T \in (0, T_0)$ such that the segment $\gamma([0, T]) \subset M$ has not self-intersection and take coordinates x_1, x_2 in a neighborhood $W \subset M$ of $\gamma([0, T])$ such that:

- $\gamma(t) = (t, 0)$ and
- $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\} \Big|_{(t,0)}$ is a orthogonal basis for $T_{(t,0)}M$, for all $t \in [0, t]$.

For each $x \in W$, let $\{dx_1, dx_2\} \subset T_x^*M$ be the dual basis for $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\} \subset T_xM$. Then (x_1, x_2, dx_1, dx_2) is a local coordinate system in a neighborhood of $\Gamma([0, T])$. In these coordinates we have $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and the Hamiltonian vector field X_H is

$$(4) \quad X_H = \sum_{i=1}^2 \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^2 \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i}.$$

Now we are going to define our perturbation space. Let $\delta, \beta : \mathbb{R} \rightarrow \mathbb{R}$ be smooth functions such that $\delta \in \{h \in C^\infty(\mathbb{R}); \text{Supp}(h) \subset (0, T)\}$ (later we will take $\delta(t)$ as a smooth approximation of a Dirac delta function) and $\beta : \mathbb{R} \rightarrow \mathbb{R}$ satisfies:

- $\text{Supp}(\beta) \subset (-\epsilon, \epsilon)$, with ϵ sufficiently small and
- $j^{k+1}(\beta)(0) = x_2^{k+1}$,

for $k > 1$. Let $\mathcal{F}^k = \mathcal{F}(W, \gamma, H, T, k)$ be the subset of functions $u : M \rightarrow \mathbb{R}$ such that, in the local coordinates (x_1, x_2) , have the form

$$u(x_1, x_2) = \delta(x_1)\beta(x_2).$$

We will consider perturbations of the kind $(H + u)$ for $u \in \mathcal{F}^k$. Then $X_{H+u} = X_H + X_u$, and locally by (4) we have

$$(5) \quad X_u = -\delta'(x_1)\beta(x_2)\frac{\partial}{\partial y_1} - \delta(x_1)\beta'(x_2)\frac{\partial}{\partial y_2} = -\delta(x_1)\beta'(x_2)\frac{\partial}{\partial y_2} + \mathcal{O}(x^{k+1}).$$

First, we choose a family of local hypersurfaces $[0, T] \mapsto \Lambda(t)$ in T^*M , given by $\Lambda(t) = [x_1 = t]$. Then $\Lambda(t)$ is transversal to $\Gamma(t)$, for all $t \in [0, T]$. For each $t \in [0, T]$, we consider the map

$$S_t(u) : (\Lambda(0), l(0)) \rightarrow (\Lambda(0), l(0))$$

defined as

$$S_t(u) = \hat{P}_t^{-1} \circ \hat{P}'_t,$$

where $\hat{P}_t : \Lambda(0) \rightarrow \Lambda(t)$ and $\hat{P}'_t : \Lambda(0) \rightarrow \Lambda(t)$ denote the Poincaré maps in an open neighborhood of $\Gamma(0) \in \Lambda(0)$ to $\Lambda(t)$ with respect to X_H and $(X_H + X_u)$ respectively. Note that the vector field X_u satisfy:

- $j^{(k-1)}(X_u)(\Gamma(t)) = 0, \forall t \in [0, T]$,
- $\Gamma(0), \Gamma(T) \notin \text{Supp}(X_u)$ and
- $X_u|_{\Lambda(t)}$ is k -tangent to $\Lambda(t)$, for all $t \in [0, T]$.

The following proposition holds for abstract vector fields satisfying the three conditions above. A proof can be seen in [KT72, section 2].

Proposition 8. *The k -jet of $S_T(u)$ at the point $\Gamma(0)$ is equal to the k -jet of the flow at time T associated to the non autonomous vector field $\hat{P}_t^*(X_u|_{\Lambda(t)})$ at the point $\Gamma(0)$.*

Let $\Sigma(t) \subset T^*M$ be the submanifold given by

$$\Sigma(t) = \Lambda(t) \cap H^{-1}(c), \quad t \in [0, T].$$

Then ω induces a symplectic structure on $\Sigma(t)$ and the restriction $\hat{P}_t|_{\Sigma(0)} : \Sigma(0) \rightarrow \Sigma(t)$ is a symplectic map for all $t \in [0, T]$. Since $\Gamma(0), \Gamma(T) \notin \text{Supp}(u)$, $\hat{P}'_T|_{\Sigma(0)} : \Sigma(0) \rightarrow \Sigma(T)$ is a symplectic map too.

Observe that $\frac{\partial H}{\partial y_1}(\Gamma(t)) \equiv 1$. Then we can parameterize $\Sigma(t)$ in terms of the coordinates x_2, y_2 , this is, for each $t \in [0, T]$ there is an open set $V_t \subset \mathbb{R}^2$ and a function $\alpha_t : V_t \rightarrow \mathbb{R}$, such that

$$\Sigma(t) = \{ (t, x_2, \alpha_t(x_2, y_2), y_2) \in \Lambda(t); (x_2, y_2) \in V_t \}.$$

Since $T\Sigma(t) \subset \text{Ker}(dx_1)$, the symplectic structure induced by ω in $\Sigma(t)$ is given by $\omega|_{\Sigma(t)} = dx_2 \wedge dy_2$. For each $u \in \mathcal{F}^k$ and $t \in [0, T]$, we consider the Hamiltonian function $K_{u,t} : \Sigma(t) \rightarrow \mathbb{R}$ given by $K_{u,t} = u|_{\Sigma(t)} = \delta(t)\beta(x_2)$ and we denote by $Y_{u,t}$ its Hamiltonian vector field. Then

$$(6) \quad j^{k+1}(K_{u,t})(\Gamma(t)) = \delta(t)x_2^{k+1},$$

and this defines a family, parameterized by t , of multiples of the polynomial $F(x_2, y_2) := x_2^{k+1}$.

Remark 9. *The submanifold $\Sigma(0)$ is not invariant by the map S_t , if $(t, x_2) \in \text{Supp}(u)$. But, by (5), the unique component of the field X_u that has the k -Jet non-vanished at $\Gamma(t)$ is the component in the direction $\frac{\partial}{\partial y_2}$ and this direction is tangent to $\Sigma(t)$ along $\Gamma(t)$ (because $dH_{\Gamma(t)}(\frac{\partial}{\partial y_2}) = \omega_{\Gamma(t)}(X_H, \frac{\partial}{\partial y_2}) \equiv 0$). Then the vector fields $Y_{u,t}$ and $X_u|_{\Sigma(t)}$ in $\Sigma(t)$ have the same k -jet along $\Gamma(t)$. Moreover, since $\hat{P}_t(\Sigma(0)) = \Sigma(t)$, we have that the non-autonomous vector field $\hat{P}_t^*(X_u|_{\Sigma(t)})$ is k -jet*

tangent to $\Sigma(0) \subset \Lambda(0)$. Therefore the submanifolds $\Sigma(0)$ and $S_t|_{\Sigma(0)}$ have a tangency of order k at $\Gamma(0)$. Then, to study the k -jet of S_t at the point $\Gamma(0)$ we can assume that S_t leaves $\Sigma(0)$ invariant for all $t \in [0, T]$.

Lemma 10. *The k -jet of $S_t|_{\Sigma(0)}$ in $\Gamma(0)$ is equal to the k -jet at $\Gamma(0)$ of the Hamiltonian flow at time t that corresponds to the non-autonomous Hamiltonian $\left[\delta(t) F \circ \left(\widehat{P}_t|_{\Sigma(0)}\right)\right]$ in $\Sigma(0)$.*

Proof: Combining the remarks 8 and 9, we conclude that the k -jet of $S_t|_{\Sigma(t)}$ is equal to the k -jet of the flow at time t associated to the field $\widehat{P}_t^*(Y_{u,t})$. On the other hand, if X denotes the Hamiltonian field for the non autonomous Hamiltonian $[K_{u,t} \circ \left(\widehat{P}_t|_{\Sigma(0)}\right)]$, then using that $\widehat{P}_t|_{\Sigma(0)} : \Sigma(0) \rightarrow \Sigma(t)$ is a symplectic map, we have:

$$\begin{aligned} \omega(X, \cdot)|_{\Sigma(0)} &= d\left(K_{u,t} \circ \left(\widehat{P}_t|_{\Sigma(0)}\right)\right) = \widehat{P}_t^*(d K_{u,t}) = \widehat{P}_t^*\omega(Y_{u,t}, \cdot)|_{\Sigma(t)} = \\ &= \omega(\widehat{P}_t^*(Y_{u,t}), \cdot)|_{\Sigma(0)}. \end{aligned}$$

And, since $\omega|_{\Sigma(0)}$ is no degenerate, we have that $X = \widehat{P}_t^*(Y_{u,t})$. Hence the k -jet of $\widehat{P}_t^*(Y_{u,t})$ in $\Gamma(0)$ is determined by the $(k+1)$ -jet of the Hamiltonian $[K_{u,t} \circ \left(\widehat{P}_t|_{\Sigma(0)}\right)]$ in $\Gamma(0)$, that, by (6), is equal to the k -jet of the Hamiltonian $\left[\delta(t)F \circ \left(\widehat{P}_t|_{\Sigma(0)}\right)\right]$. This completes the proof. \square

Remark 11. Recall that $J_s^k(1)$ is a Lie Group with the group structure defined by $j^k(f) \cdot j^k(g) = j^k(f \circ g)$. Let $\mathfrak{J}_s^k(1)$ be the space of the k -jets in $0 \in \mathbb{R}^2$ of the symplectic vector fields in $(\mathbb{R}^2, dx_1 \wedge dx_2)$ that are zero in the origin. We define the bracket $[\cdot, \cdot]^k : \mathfrak{J}_s^k(1) \times \mathfrak{J}_s^k(1) \rightarrow \mathfrak{J}_s^k(1)$ by $[j^k(X), j^k(Y)]^k = -j^k([X, Y])$. Since X, Y are zero in the origin, $[\cdot, \cdot]^k$ depends only on the k -jets of X and Y . Then $[\cdot, \cdot]^k$ defines a Lie algebra structure in $\mathfrak{J}_s^k(1)$. Moreover, $\mathfrak{J}_s^k(1)$ is the Lie algebra of $J_s^k(1)$ and the exponential map $\exp : \mathfrak{J}_s^k(1) \rightarrow J_s^k(1)$ is given by $\exp(t j^k(X)) = j^k(\psi_t)$, where ψ_t is the local flow associated to X . For more details and proofs, see [KMS93, §IV]

Lemma 12. *Let $\pi_k : j_s^k(1) \rightarrow j_s^{k-1}(1)$ be the canonical projection. Given an integer $k \geq 2$, consider the map*

$$\begin{aligned} \mathcal{S}_T^k : \mathcal{F}^k &\longrightarrow \text{Ker}(\pi_k) \subset J_s^k(1) \\ u &\longmapsto j^k(S_T(u)|_{\Sigma(0)})(\Gamma(0)). \end{aligned}$$

If the one parameter family $[0, T] \rightarrow d_{\Gamma(0)}\widehat{P}_t|_{\Sigma(0)} \subset j_s^1(1) = Sp(1)$, is $(k+1)$ -general for some $k > 1$, then the map \mathcal{S}_T^k is a local submersion in neighborhood of $0 \in \mathcal{F}^k$.

Proof. Since $d_{\Gamma(0)}\widehat{P}_t|_{\Sigma(0)}$ is $(k+1)$ -general, for $0 \leq t \leq T$, there are $t_1, \dots, t_{k+1} \in (0, T)$, such that $\left\{F(x_2, y_2), F(d\widehat{P}_{t_1}(x_2, y_2)), \dots, F(d\widehat{P}_{t_{k+1}}(x_2, y_2))\right\}$ is a basis for $\mathbb{R}[x_2, y_2]^{k+1}$, where $F(x_2, y_2) = x_2^{k+1} \in \mathbb{R}[x_2, y_2]^{k+1}$. For each $0 \leq i \leq k+1$ and $\lambda > 0$ sufficiently small, let $\delta_\lambda(t_i) : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ approximation of the Dirac delta function at the point t_i with support in the interval $[t_i - \lambda, t_i + \lambda]$. We consider $u_i = u_i(\lambda) = \delta_\lambda(t_i)\beta(x_2) \in \mathcal{F}^k$, for $i \in \{0, \dots, k+1\}$. By Lemma 10 and the properties of the exponential map, as defined in the Remark 11, we have:

$$D_0 \mathcal{S}_t^k \cdot (u_i) = \left. \frac{\partial}{\partial s} \right|_{s=0} \mathcal{S}_t^k(s u_i) = \left. \frac{\partial}{\partial s} \right|_{s=0} \exp(t j^k(s X_i)),$$

where X_i denotes the Hamiltonian field in $\Sigma(0)$ corresponding to the non autonomous Hamiltonian $[\delta_\lambda(t_i)(t)F \circ \left(\widehat{P}_t|_{\Sigma(0)}\right)]$. Computing the derivative with respect to t in the above equality, we obtain:

$$\frac{d}{dt}(D_0 \mathcal{S}_t^k \cdot (u_i)) = \left. \frac{\partial}{\partial s} \right|_{s=0} \left(\frac{\partial}{\partial t} \exp(t j^k(s X_i)) \right) =$$

$$= \frac{\partial}{\partial s} \Big|_{s=0} \left(d_{(t, j^k(s, X_i))} \exp \cdot j^k(s, X_i) \right) = \frac{\partial}{\partial s} \Big|_{s=0} j^k(s, X_i) = j^k(X_i).$$

Then

$$(7) \quad D_0 \mathcal{S}_T^k \cdot (u_i) = \int_0^T j^k(X_i) dt.$$

By definition of X_i and (7), we have that λ converges to 0 then $D_0 \mathcal{S}_T^k \cdot (u_i)$ converges to the k -jet in $\Gamma(0)$ of the Hamiltonian field in $\Sigma(0)$ correspondent to the autonomous Hamiltonian $H_i = [F \circ (\hat{P}_{t_i}|_{\Sigma(0)})]$. Computing the $(k+1)$ -jet of H_i in $\Gamma(0)$, we obtain:

$$j^{k+1}(H_i) = [F \circ (d_{\Gamma(0)} \hat{P}_{t_i}|_{\Sigma(0)})].$$

Since $\{F \circ (d_{\Gamma(0)} \hat{P}_{t_i}|_{\Sigma(0)})\}_{0 \leq i \leq k+1}$ is a basis for $\mathcal{R}[x_2, y_2]_{k+1}$, we have that for λ sufficiently small $\{D_0 \mathcal{S}_T^k \cdot (\xi_i)\}_{0 \leq i \leq k+1}$ is a basis for the Lie Algebra of the Lie subgroup $\text{Ker}(\pi_k)$. Hence the map \mathcal{S}_T^k is a local submersion. \square

2.3. Proof of the Theorem 1. Let $\Gamma(t) = (\gamma(t), p(t))$ be a closed orbit of the Hamiltonian flow ψ_t^H of minimal period $T_0 > 0$. Since the number of self-intersection points is finite, we can choose $T \in (0, T_0]$, such that the segment $\gamma([0, T])$ does not contain self-intersection points of the curve γ and a tubular neighborhood $W \subset M$ of $\gamma([0, T])$, sufficiently small, such that $W \cap \gamma = \gamma([0, T])$. By this, we can choose a local coordinates system x_1, x_2 in W and a family of local transversal sections $\Sigma(t) = \Lambda(t) \cap H^{-1}(c) \subset T^*M$, and maps $\hat{P}_t : \Lambda(0) \rightarrow \Lambda(t)$, as in subsection 2.2. Then $P_t := P(\Gamma(0), \Sigma(0), \Sigma(t), H) = \hat{P}_t|_{\Sigma(0)}$. By Lemma 7, there exists a smooth potential u with C^∞ -norm arbitrarily small, such that the correspondent one parameter family $[0, T] \mapsto d_{\Gamma(0)} P_t$ is s-general, for any $s = 2, 3, \dots, k+1$.

We set $\mathcal{F}^i = \mathcal{F}^i(W, \gamma, H, T, i) \subset C^\infty(M)$, $2 \leq i \leq k$, as in section 2, and

$$\mathcal{F} = \mathcal{F}(W, \gamma, T) = \left\{ u \in C^\infty(M); u|_{\gamma([0, T])} \equiv 0 \text{ and } \text{Supp}(u) \subset W \right\}.$$

It is easy to see that $\mathcal{F}^i \subset \mathcal{F}$ for all $i \in \{1, \dots, k\}$. We define the map:

$$\begin{aligned} \mathcal{S} : \mathcal{F} &\rightarrow J_s^k(1) \\ u &\mapsto j^k(S(u))(\Gamma(0)) \end{aligned}$$

where $S(u) = P(\Sigma(0), \Sigma(0), H)^{-1} \circ P(\Sigma(0), \Sigma(0), H + u)$.

Remark 13. Since $\text{Supp}(u) \subset W$ for all $u \in \mathcal{F}$ and

$$P(\Sigma(0), \Sigma(0), q) = P(\Sigma(0), \Sigma(T), q) \circ P(\Sigma(T), \Sigma(T_0), q),$$

with $q = \{H, H + u\}$, its follows that $S(u) = S_T(u) = P_T(H)^{-1} \circ P_T(H + u)$.

By Remark 6 and Lemma 12, we have that each $r = 2, \dots, k$, the map \mathcal{S}_T^r is an open map in a neighborhood of 0 in \mathcal{F} . Since $j^k(P(\theta, \Sigma, H)) \in \overline{Q}$, the openness of the maps \mathcal{S}_T^r (for $r = 2, \dots, k$) in a neighborhood of zero implies that there exists $u \in \mathcal{F}$ arbitrarily C^∞ -close to zero, such that the k -jet of $S_T(u)$ is a element of the set Q . By remark 13, its proves the theorem.

3. PROOF OF THEOREM 2

Let Q be an open, dense and invariant subset of $J_s^k(1)$ and $c \in \mathcal{R}$ (fixed). As in the Kupka-Smale Theorem, for each $n \in \mathbb{N}$, let $\mathcal{O}(c, n) \subset C^\infty(M)$ be such that $E_{L-u}^{-1}(c)$ are regular and every closed orbit of the flow ϕ_t^{L-u} in $E_{L-u}^{-1}(c)$ with period $\leq n$ are nondegenerate, for all $u \in \mathcal{O}(c, n)$. Then $\mathcal{O}(c, n)$ is open and dense subset of $C^\infty(M)$ with the C^∞ -topology (see Lemma 3.3 in [Oli08]). Let

$\mathcal{G}(n) \subset \mathcal{O}(c, n)$ be the set of C^∞ -potentials $u : M \rightarrow \mathbb{R}$ such that the k -jet of the Poincaré map of every closed orbit of ϕ_t^{L-u} contained in $E_{L-u}^{-1}(c)$ and with period $\leq n$ belongs to Q . Since the set of periodic orbits of ϕ_t^{L-u} in $E_{L-u}^{-1}(c)$ with period $\leq n$ is finite, for all $u \in \mathcal{G}(n)$, and by continuity of the Poincaré map, we have that $\mathcal{G}(n)$ is open. By Theorem 1, $\mathcal{G}(n)$ is also C^∞ -dense subset of $C^\infty(M)$. Therefore

$$\mathcal{G} = \bigcap_{n \in \mathbb{N}} \mathcal{G}(n)$$

is the residual subset that we are looking for. \square

4. LAGRANGIAN FLOWS WITH A NON-HYPERBOLIC CLOSED ORBIT.

In this section we prove the Proposition 3. For this, we follow the strategy in the proof of an analogous result for geodesic flows on S^2 [CBP02, Prop. 3.3]. In [Mir07, Th. 1.1], we used similar arguments for magnetic flows on surfaces with a non-hyperbolic closed orbit.

Let us recall the Birkhoff's normal form, for a proof see [SM95, p 222]

Theorem 14. *Let f be a C^4 diffeomorphism defined in a neighborhood of $0 \in \mathbb{R}^2$ such that f preserves the area form $dx \wedge dy$ and $f(0) = 0$. Suppose that the eigenvalues of $d_0 f$ satisfy: $|\lambda| = 1$ and $\lambda^n \neq 1$, for all $n \in \{1, \dots, 4\}$. Then there exists a C^4 diffeomorphism h , defined in a neighborhood of 0 such that $h(0) = 0$, h preserves the form $dx \wedge dy$ and, in polar coordinates (r, θ) , we have:*

$$h^{-1} \circ f \circ h(r, \theta) = (r, \theta + \alpha + \beta r^2) + \mathcal{O}(r^4).$$

Moreover, the property of $\beta \neq 0$ uniquely depends of f .

We say that a homeomorphism $f : [a, b] \times S^1 \rightarrow [a, b] \times S^1$ is a *twist map* if for all $\theta \in S^1$ the function $[a, b] \mapsto \pi_2 \circ f(\cdot, \theta) \in S^1$ is strictly monotonous. Observe that if the coefficient $\beta = \beta(f)$ in the normal form is not zero, then for $|r| \leq \epsilon$, with ϵ small enough, f is conjugated to a twist map in $[0, \epsilon] \times S^1$.

We shall use the following result:

Proposition 15 (Le Calvez [LC91, Remarques pg.26]). *Let f be a diffeomorphism of the annulus $\mathbb{R} \times S^1$ such that it is a twist map, it is area preserving, the form $f^*(R d\theta) - R d\theta$ is exact and*

- (i) *If x is a periodic point for f and q is its period, the eigenvalues of $d_x f^q$ are not roots of unity.*
- (ii) *The stable and unstable manifolds of hyperbolic periodic orbits of f intersect transversally (i.e. whenever they meet, they meet transversally).*

Then f has periodic orbits with homoclinic points.

We are now ready to show the Proposition 3.

4.1. Proof of Proposition 3. Let $\theta_t = \phi_t^L(\theta)$ a non-hyperbolic closed orbit of minimal period $T > 0$, contained in $E_L^{-1}(c)$. Let $P(\theta, \Sigma, L)$ the Poincaré map for a local transversal section $\Sigma \subset E_L^{-1}(c)$ in θ . Since θ_t is non-hyperbolic, we have that the eigenvalues of $d_\theta P$ are of the form $e^{\pm 2\pi i \alpha}$, with $\alpha \in [0, 1)$.

We consider the subset $Q \subset j_s^3(1)$, defined as:

$$Q = \left\{ \sigma \circ f_{\alpha, \beta} \circ \sigma^{-1} ; \sigma \in j_s^3(1), \beta > 0, \text{ and } \alpha \notin \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{4}{3}, \frac{3}{2} \right\} \right\},$$

where $f_{\alpha, \beta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by $f_{\alpha, \beta}(r, \theta) = (r, \theta + \alpha + \beta r^2) + \mathcal{O}(r^4)$, in polar coordinates. By the Birkhoff's normal form, the subset $Q \subset j_s^3(1)$ is open and invariant. Since the orbit θ_t is non-hyperbolic, we have that $j^3(P(\theta, \Sigma, L)) \in \overline{Q}$. So, the Theorem 1 implies the existence of $u : M \rightarrow \mathbb{R}$

arbitrarily C^∞ -close to $0 \in C^\infty(M)$ such that: θ_t is a closed orbit of same period for the flow ϕ_t^{L-u} and $j^3(P(\theta, \Sigma, L - u)) \in Q$.

Observe that θ_t is elliptic for the perturbed flow ϕ_t^{L-u} . Therefore, there is a neighborhood $\mathcal{U} \subset C^\infty(M)$ of u , such that for all $\bar{u} \in \mathcal{U}$, the flow $\phi_t^{L-\bar{u}}|_{E_L^{-1}(c)}$ has an elliptic closed orbit $\bar{\theta}_t$ close to θ_t that we call *analytic continuation of θ_t* . Since Q is open, if the neighborhood \mathcal{U} is small enough, we can assume that $j^3(P(\bar{\theta}, \Sigma, L - \bar{u})) \in Q$, for all $\bar{u} \in \mathcal{U}$.

By Kupka-Smale Theorem, we can approximate u for $\bar{u} \in \mathcal{U}$ such that $\bar{\theta}_t$ is the analytic continuation of θ_t , $f = P(\bar{\theta}, \Sigma, L - \bar{u})$ satisfies the conditions (i) and (ii) of the Proposition 15, $j^3(f) \in Q$ and, via Darboux coordinates, f is a diffeomorphism in a neighborhood of $0 \in \mathbb{R}^2$ that preserves the area form $dx \wedge dy$.

By definition of Q , the map f is conjugated to a twist map $f_0 = hfh^{-1}$, in polar coordinates. In order to apply the Proposition 15, we need to do a change of coordinates which transforms f_0 in a twist map $T : \mathbb{R}^+ \times S^1 \rightarrow \mathbb{R}^+ \times S^1$, such that the 1-form $T^*(Rd\theta) - Rd\theta$ is exact. Then the existence of a homoclinic orbit implies the existence of a non-trivial hyperbolic set.

In fact, we consider the following maps:

$$\begin{array}{ccccc} (x, y) & \longrightarrow & (r, \theta) & \longrightarrow & (\frac{1}{2}r^2, \theta) = (R, \theta) \\ & & & & \\ \mathbb{D} & \xrightarrow{P} & \mathbb{R}^+ \times S^1 & \longrightarrow & \mathbb{R}^+ \times S^1 \\ f \downarrow & & f_0 \downarrow & & \downarrow T \\ \mathbb{D} & \longrightarrow & \mathbb{R}^+ \times S^1 & \longrightarrow & \mathbb{R}^+ \times S^1 \end{array}$$

where $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, $P^{-1}(r, \theta) = (r \cos \theta, r \sin \theta)$. Let $G(x, y) = (\frac{1}{2}r^2, \theta) = (R, \theta)$. Then $\lambda := G^*(R d\theta) = \frac{1}{2}(x dy - y dx)$. Observe that $d\lambda = dx \wedge dy$ is the area form \mathbb{D} . Since that \mathbb{D} is contractible, we have that $f_0^*(\lambda) - \lambda$ is exact. Therefore $T^*(R d\theta) - R d\theta$ is exact. Since $R(r) = \frac{1}{2} r^2$ strictly increasing on $r > 0$, T is a twist map if and only if f_0 is a twist map. \square

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